Numerical challenges to decomposition of real algebraic surfaces

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Solving Polynomial Systems

Varieties and Components

Curves: $k = 1$

Surfaces: $k = 2$

Singular Curves

Conclusion
Polynomial Systems

Polynomials are everywhere!

- chemical reaction networks
- kinematics
- dynamical systems
- optimization
- many more
example $p(z)$

\begin{align*}
  x_0 + x_1 + x_2 + y_1 + y_2 + y_3 + y_4 - k_1 &= 0, \\
  e + y_1 + y_2 - k_2 &= 0, \\
  f + y_3 + y_4 - k_3 &= 0, \\
  -a_1 x_0 e + b_1 y_1 + c_4 y_4 &= 0, \\
  c_2 y_2 - a_3 x_2 f + b_3 y_3 &= 0, \\
  a_1 x_0 e - (b_1 + c_1) y_1 &= 0, \\
  a_2 x_1 e - (b_2 + c_2) y_2 &= 0, \\
  a_3 x_2 f - (b_3 + c_3) y_3 &= 0, \\
  a_4 x_1 f - (b_4 + c_4) y_4 &= 0.
\end{align*}

How to solve a system $p(z) = 0$? That’s hard!
How to solve $p(z) = 0$

There are many ways to ‘solve’ a polynomial system:

▶ If linear, invert the matrix. This is just linear algebra. Otherwise...

▶ Symbolically, the main method to use is Gröbner basis. Available in Macaulay2, Singular, Maple, etc. Work exactly over $\mathbb{C}$ to find an ideal basis for the system $p$. ⇒ Result is a set of polynomials.

▶ Numerically, homotopy continuation methods find approximate solutions. Available in HOM4PS, Bertini, and others ⇒ Result is a set of points.
Homotopy Methods

$f(z) \quad H(z,t) \quad g(z)$

paths diverge convergent
paths

$\infty$

$C^N$

$\Delta t$

$t = 0 \quad t = 1$
Start Systems

There are lots of ways to make start system $g$.

- Roots of unity - $g_i = z_i^{d_i} - 1$
- Total degree - $g_i = \prod_{j=1}^{d_i} \mathcal{L}_j(\vec{x})$
- Multi-homogeneous - $g_i = \prod_j \mathcal{L}(\vec{x}_{\sigma_k})$
- Polyhedral - $g_i$ determined by the Newton Polytope

The canonical linear homotopy:

$$H(z, t) = t \cdot f(z) + \gamma (1 - t) \cdot g(z) = 0$$
Path Tracking

Given a homotopy $H(z, t)$, we wish to move in $t$ from $t_1$ to $t_0$. How? By solving the Davidenko differential equation:

$$
\frac{\partial H(z(t), t)}{\partial t} + \sum_{i=1}^{N} \frac{\partial H(z(t), t)}{\partial z_i} \frac{dz_i(t)}{dt} = 0
$$

which more or less amounts to solving

$$
\frac{dz(t)}{dt} = -[JH(z(t), t)]^{-1} \frac{\partial H(z(t), t)}{\partial t}.
$$
Path Tracking

We can solve this diff-eq by discrete approximation and an Euler prediction step:

\[ p_{i+1} = p_i - [JH(p_i, t_i)]^{-1} \frac{\partial H(p_i, t_i)}{\partial t} \Delta t_i \]

followed by a Newton correction step:

\[ z_{i+1} = z_i - [JH(z_i, t_{i+1})]^{-1} H(z_i, t_i) \]
Difficult 1

Our first numerical difficulties:

- How to choose $\Delta t_i$?
- Is $[JH(p_i, t_i)]^{-1}$ and $[JH(z_i, t_{i+1})]^{-1}$ always computable?

Solutions to these problems:

- $\Delta t_i$ is chosen adaptively, and we are permitted to step tentatively.
- Use adaptive multiple precision, to overcome the loss of precision through inversion. 
  $\Rightarrow$ you lose precision when inverting according to the condition number, the ratio of the two extreme singular values.
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Variety

The *zero-set* of a system of polynomials is the *variety*. Varieties consist of *components*, which have meta-data attached to them:

- dimension,
- degree,
- and others.

One primary goal of algebraic geometry is to discover, for $p(z)$, what is the structure of its variety $\mathcal{V}(p)$?
Witness Sets

Witness set - Numerical Algebraic Geometry’s representation of a positive dimensional component $C$ over $\mathbb{C}^N$.

Consists of three things:

- points $x$ – generic, nonsingular. $\# = \text{degree}(C)$.
- linears $L$ – generic. $\# = \text{dim}(C)$.
- system $f$ – possibly randomized, $Jf$ is full rank on each $x$.

Positive dimensional $\mathbb{C}$ components are nice - degree and structure are same almost everywhere.

$\mathbb{R}$ not so much...
Real Posdim Components

Bertini_real – Numerical Cellular Decomposition

Setup

- Let $f$ be a polynomial system with $\mathbb{R}$ coefficients, and $f : \mathbb{C}^N \to \mathbb{C}^n$.
- Let $\mathcal{V}(f)$ be the variety of $f$.
- Consider $C \subseteq \mathcal{V}(f)$ be a component of dimension $k$.

Our objective is to decompose the real part of $C$; i.e., $C \cap \mathbb{R}^N$ or $\mathcal{V}_{\mathbb{R}}(f)$. 
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Curves: $k = 1$
Curves: $k = 1$

Curve Cell
Curves

1. Find critical points
2. Intersect with sphere
3. Slice
4. Connect the dots
5. Merge
6. Refine

[Lu, Bates, Sommese, Wampler, 2006]
Curves

twisted cubic

\[ f(x, y, z) = \begin{bmatrix} y - x^2 \\ z - x^3 \\ y^2 - xz \end{bmatrix} \]
Curves

Burmester 3-3 curve, dimension 14.
In 2d projection, of degree 128.
Critical points of curves

Computing critical points of curves is easy.\[^{\text{citation needed}}\]

Since $f$ defines a dimension-one component, the working witness set comes with one random linear:

\[
\begin{bmatrix}
  f \\
  \mathcal{L}_1
\end{bmatrix}
\]

Then we use regeneration to solve the system

\[
\begin{bmatrix}
  f \\
  \det \left( \begin{bmatrix} J_f & J_{\pi_1} \end{bmatrix} \right)
\end{bmatrix}
\]

[Hauserstein, Sommese, Wampler, 2009]
Numerical Challenge 2

The second challenge –

- critical points are often singular.
That is, the Jacobian is singular! The condition number is infinite!

The solution –

- compute nullvectors simultaneously to the $z$ variables of interest:

$$
\begin{bmatrix}
    f(x) \\
    v^T \cdot \begin{bmatrix}
        Jf(x) \\
        J\pi_1 \\
        \text{patch}_v
    \end{bmatrix}
\end{bmatrix}
$$
Solving Polynomial Systems

Varieties and Components

Curves: \( k = 1 \)

Surfaces: \( k = 2 \)

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Conclusion
Surfaces

$k = 2$
Surface Cell

Surfaces: $k = 2$
Surface Decomposition

1. Decompose critical curve
2. Decompose singular curves
3. Intersect with sphere
4. Slice
5. Connect the dots
6. Refine

[Besana, Di Rocco, Hauenstein, Sommese, Wampler, 2013]
Surface examples

\[
f(x, y, z) = (x^2 + y^2 + z^2 + 2^2 - \frac{1}{2})^2 - 16(x^2 + y^2)
\]
Surface examples

distel, unrefined

\[ f(x, y, z) = x^2 + y^2 + z^2 + 1000(x^2 + y^2)(x^2 + z^2)(y^2 + z^2) - 1 \]

distel, refined
Surfaces examples

solitude

\[ f(x, y, z) = x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \]

klein

\[ f(x, y, z, w) = \begin{bmatrix} w^2 + x^2 + y^2 + z^2 - 1 \\ 2wyz - x(y^2 + z^2) \end{bmatrix} \]
Fivebar mechanism kinematics
Six-dimensional trigonometric system,
passed through an atan2 projection
Midtracking

Simultaneously track all three systems
Challenge 3

Our third challenge:

- Midtracking tracks *three* systems simultaneously,
- The two boundary systems are often critical or singular curves,
- The endpoints on the boundary are often singular.

Solution:

- Rely on *adaptive multiple precision* and endgames to keep things nice.
Crit points of surfaces

Computing critical points of surfaces is hard.

- Surfaces have *critical curves*.
- Surfaces also have critical points themselves.
Numerical Challenge 4

Using the *determinantal* form of the criticality conditions:

witness set:

\[
f_{\text{crit\_curve}} = \begin{vmatrix}
  f \\
  Jf \\
  J\pi_1 \\
  J\pi_2 \\
  L_1
\end{vmatrix}
\]

\[
f_{\text{crit\_crit}} = \begin{vmatrix}
  f \\
  Jf \\
  J\pi_1 \\
  J\pi_2 \\
  \det \left( \begin{vmatrix}
    f \\
    Jf \\
    J\pi_1 \\
    J\pi_2
  \end{vmatrix} \right) \\
  \det \left( \begin{vmatrix}
    Jf \\
    J\pi_1 \\
    J\pi_2
  \end{vmatrix} \right) \\
  \det \left( \begin{vmatrix}
    Jf \\
    J\pi_1 \\
    J\pi_2
  \end{vmatrix} \right) \\
  \text{patch}_v
\end{vmatrix}
\]

- Determinant operation produces high degree polynomials, contributing to numerical issues.
- Using this formulation for dimension 3 decompositions will be even worse w.r.t. degree and computational complexity.
- Fortunately, we can still use the nullspace method for finding critical points of this curve.
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Example - Solitude
How to find singular curves

1. Compute witness set for critical curve.
2. Separate singular points from nonsingular.
   - Nonsingular $\rightarrow$ critical curve.
   - Singular $\rightarrow$ singular curve(s).
3. Separate singular witness points by deflation sequence.
4. Decompose each singular curve.
Practical Challenge 5 - Deflation

Deflation

- Making the untrackable, trackable.
- Removing singularity of points.

Deflation:

- Determinental - adds no variables, but \textbf{lots} of functions

\[ g_{\det} = \begin{bmatrix} f(x) \\ \det J_{\sigma_1} f(x) \\ \vdots \\ \det J_{\sigma_m} f(x) \end{bmatrix} \]

[Hauenstein, Wampler '13]
Example - Solitude

Solitude:

\[ f(x, y, z) = x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \]

\[ D^1 = \begin{bmatrix}
  x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \\
  y^2/4 - (xz^2)/2 + (xyz)/2 \\
  (xy)/2 + (x^2z)/4 + (3y^2z)/4 + (3y^2)/4 \\
  (x^2y)/4 - (x^2z)/2 + y^3/4
\end{bmatrix} \]
\begin{equation}
\mathcal{D}^2 = \begin{bmatrix}
x^2yz + xy^2 + y^3 + y^3z - x^2z^2 \\
y^2/4 - (xz^2)/2 + (xyz)/2 \\
(xy)/2 + (x^2z)/4 + (3yz^2)/4 + (3y^2)/4 \\
(x^2y)/4 - (x^2z)/2 + y^3/4 \\
(y^2z^2)/8 - (x^2z^3)/24 + (y^2z^3)/8 - (yz^2)/8 - (y^3z)/8 + y^3/24 + (xy^2z)/24 + (x^2yz^2)/24 \\
(x^2z^3)/12 + (y^3z^2)/24 + (xy^3)/24 - (x^4z)/24 - (xy^2z)/12 - (x^2yz^2)/8 + (x^2y^2z)/24 \\
(x^2y^2)/24 + (xy^3)/8 - y^4/24 - (xy^2z)/4 - (x^2yz)/12 + (y^3z)/12 - (xy^2z^2)/4 \\
-(x(x^2z^2 - 3y^2z^2 + 2xz^2 + 6yz^2 - 3y^2z + 6yz^3 + y^2 + xyz))/24 \\
(y^3)/12 - (x^2y^2)/16 + (x^3z)/12 + (x^4z)/48 + (y^4z)/16 + y^4/16 + (x^2yz)/4 + (x^2yz^2)/4 \\
(x^3z^2)/24 - (x^2y^2)/16 - (xy^2)/8 + y^4/16 + (xy^2z)/4 + (x^2yz)/6 + (x^2y^2z)/8 \\
-(x(x^2y^2 + 2x^2z^2 + xy^2 - 2y^3z + y^4 - 2x^2yz))/24 \\
-(y(4x^2y^2 + 6x^2y + 4x^3 + x^4 + 3y^4))/48 \\
(y^2z^2)/12 - (x^2z^2)/36 - (x^2z)/36 - (yz^2)/12 + (y^2z)/12 - (yz^3)/12 - y^2/36 - (xyz)/36 \\
(x^2z^3)/24 - (y^2z^2)/24 - (x^2z)/24 + (y^3z)/24 - (x^2yz)/72 + (xyz)/18 \\
(x^2z)/18 - (x^2y)/72 - (xy^2)/12 + (x^3z)/72 + y^3/24 + (xyz^2)/6 - (x^2yz)/24 + (xyz)/6 \\
(x^2z^2)/24 - (y^2z^2)/24 - (x^2y^2)/24 - (x^2z)/24 + (y^3z)/24 - (x^2yz)/72 + (xyz)/18 \\
-(x^2(y^2 - 3yz + 3z^2))/36 \\
-(xy(2x - 6yz + x^2 + 3y^2))/72 \\
(x^2z)/18 - (x^2y)/72 - (xy^2)/12 + (x^3z)/72 + y^3/24 + (xyz^2)/6 - (x^2yz)/24 + (xyz)/6 \\
-(xy(2x - 6yz + x^2 + 3y^2))/72 \\
-(x^2y^2)/24 - (x^2y)/12 - x^3/36 - x^4/144 - y^4/16 - (x^2yz)/12
\end{bmatrix}
\end{equation}
3D Printing

1. Run Bertini
2. Run Bertini_real
3. Refine
4. Process into .stl
5. Thicken surface
6. Print

<table>
<thead>
<tr>
<th>dimension</th>
<th>components</th>
<th>classified</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>4</td>
</tr>
</tbody>
</table>

****** Witness Set Decomposition ******

****** Decomposition by Degree ******
Dimension 2: 1 classified component
degree 4: 1 component
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Last Notes

- There are lots of ways to get ahold of algebraic varieties,
- I discussed *numerical* methods here,
- The fundamental computed unit at every stage is a point,
- Inversion of matrices incurs *loss of precision*,
- Techniques to overcome numerical challenges involve:
  - Solving a different problem (*reformulation*),
  - Using longer numbers (*adaptive multiple precision*),
  - Using fancy techniques at the end of a path (*endgames*),
Thank you for your kind attention.
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Curve Merging

Homotope an old midpoint

Delete all red objects

Combined merge -- merges multiple edges simultaneously

\[ p_{\text{new}} = \frac{p_{\text{left}} + p_{\text{right}}}{2} \]

\[(1 - t) \left[ \frac{f(x)}{\pi_0(x) - p_{\text{new}}} \right] + t \left[ \frac{f(x)}{\pi_0(x) - p_{\text{old}}} \right] = 0 \]
Curve Sampling

Fixed-number sampling

Uneven in space

Discretize so has $\mu$ total points

Perfectly uniform in projection value

Adaptive sampling

More uniform in space

Bisect intervals until distance $< \varepsilon$

Non-uniform in projection value
Surface Refinement

Raw face and edges

Top
Right
Midpoint
Bottom
Left

Refine the edges

Add ribs

Triangulate