The complete solution of Alt-Burmester synthesis problems for four-bar linkages

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Outline

Setting up the problem

Solution

Examples

Conclusion
Four-bar linkages
Kinematic Synthesis

Design a mechanism with a given motion:

1. Select particular design constraints representing requirements
2. Phrase as a polynomial system
3. Solve the problem
4. Select a computed mechanism
5. Build it
1. Select design constraints

- $M = \#$ poses yellow squares
- $N = \#$ points yellow dots
2. Phrase as a polynomial system

for $j = 2, \ldots, M + N$

$$L_{1j} \bar{L}_{1j} - L_{11} \bar{L}_{11} = 0. \quad (1)$$

for $j = 1, \ldots, M + N$

$$L_{2j} = \Theta_j z_2 + D_j - G_2, \quad (2)$$

$$\bar{L}_{2j} = \bar{\Theta}_j \bar{z}_2 + \bar{D}_j - \bar{G}_2, \quad (3)$$

along with, for $j = 2, \ldots, M + N$,

$$L_{2j} \bar{L}_{2j} - L_{21} \bar{L}_{21} = 0. \quad (4)$$

Gives “the $(M, N)$ problem”
3. Solve the problem

The main numerical method used to solve polynomial systems is *homotopy continuation*

1. Form a homotopy.
2. Track from start time and point to target.
4. Select computed mechanism

- To build a mechanism, you have to have its design parameters
- The design parameters are the solution to the polynomial system
- Depending on $M, N$, the dimension $D$ of the set varies:

  $$D = 10 - 2M - N.$$  

- For some $M, N$, $D = 0 \Rightarrow \exists$ finitely many solutions.
- For others, $D > 0 \Rightarrow \infty$ many solutions
5. Build it!

Now the maker in you comes out. Have design parameters, just need to

- get some bearings
- get some motors
- get a tool for the end effector
- write software to generate 3D model from design parameters to fit the above
- print
- assemble
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Complete solution

What do I mean by *complete solution*?

- For each possible \((M, N)\) problem describe
  - The dimension of the set
  - The degree of a generic solution set, under two projections

- and additionally, for some \((M, N)\),
  - The number of critical points of 1-dimensional sets
# Degrees

**Table:** Dimension and degree of Alt-Burmester solution sets, projected onto all eight natural variables.

<table>
<thead>
<tr>
<th>$N$</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<td>24</td>
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<td>6</td>
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<td>4</td>
<td>134</td>
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Table: Dimension and degree of Alt-Burmester solution sets, projected onto the center point coordinates, \((G_1, \overline{G}_1)\).

<table>
<thead>
<tr>
<th>(M)</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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</table>

* indicates closure of projection filled entire space
# Numbers of critical points

## Table: Numbers of critical points for all one-dimensional Alt-Burmester problems.

<table>
<thead>
<tr>
<th>( (M, N) )</th>
<th>( 8 ) natural vars</th>
<th>( (G_1, \overline{G}_1) )</th>
<th>( (z_1, \overline{z}_1) )</th>
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<td>( (1, 7) )</td>
<td>55676</td>
<td>4168</td>
<td>38740</td>
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<td>( (2, 5) )</td>
<td>11228</td>
<td>988</td>
<td>8084</td>
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<td>( (3, 3) )</td>
<td>1440</td>
<td>144</td>
<td>972</td>
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<tr>
<td>( (4, 1) )</td>
<td>152</td>
<td>16</td>
<td>92</td>
</tr>
<tr>
<td>( )</td>
<td>nonsing.</td>
<td>sing.</td>
<td>nonsing.</td>
</tr>
</tbody>
</table>
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A (4,2) problem – point solutions
Previous (4,2) solution, on a curve of (4,0) solutions
A (4,1) problem – a curve of solutions

discontinuous solution set – no mechanism on the missing section satisfies the constraints
A \((3,3)\) problem – a curve of solutions

12 asymptotes to infinity – correspond to slider mechanisms
A $(3,3)$ problem – zoom 1

a complicated structure emerges
A (3,3) problem – zoom 2

what a tangled mess, with three apparent nodes
A (3,3) problem – the pole triangle

The nodes are on the vertices of the pole triangle!
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Conclusions

The problem of four-bar linkage kinematic synthesis is solved

- Degree and dimension for solution set for all \((M, N)\) problems
- How many critical points there can be for curve solutions
- How to compute a mechanism for any \((M, N)\) problem

Future work:

- Tackle spatial mechanisms
- automatic 3D-printing of a target mechanism
- Optimization on positive-dimensional solution sets
Thank you for your kind attention!

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