Decomposing solution sets of polynomial systems using derivatives

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Abstract. A core computation in numerical algebraic geometry is the decomposition of the solution set of a system of polynomial equations into irreducible components, called the numerical irreducible decomposition. One approach to validate a decomposition is what has come to be known as the "trace test." This test, described by Sommese, Verschelde, and Wampler in 2002, relies upon path tracking and hence could be called the "tracking trace test." We present a new approach which replaces path tracking with local computations involving derivatives, called a "local trace test." We conclude by demonstrating this local approach with examples from kinematics and tensor decomposition.

Keywords: Numerical algebraic geometry, trace test, numerical irreducible decomposition

1 Introduction

Numerical algebraic geometry uses numerical methods to compute and manipulate the solution set to a given system of polynomial equations. Such a solution set can be decomposed into finitely many components yielding the irreducible decomposition. In numerical algebraic geometry, irreducible components are represented via a witness set with a numerical irreducible decomposition consisting of a witness set for each irreducible component. See [3, 12] for a general overview of witness sets and computing a numerical irreducible decomposition.

The focus of this article is the pure-dimensional decomposition step in computing a numerical irreducible decomposition. Let X be a pure k-dimensional component of the solution set of f, namely $\mathcal{V}(f) = \{x \mid f(x) = 0\}$, and let $\mathcal{L}$ be a general linear space of codimension k. That is, X is a union of irreducible components of $\mathcal{V}(f)$ each having dimension k, say $X = X_1 \cup \cdots \cup X_m$. Given the finitely many points $W = X \cap \mathcal{L}$, called a witness point set for X, the pure-dimensional decomposition step partitions W into $X_1 \cap \mathcal{L}, \ldots, X_m \cap \mathcal{L}$ yielding witness point sets for the irreducible components of X.

There are two tools commonly used for pure-dimensional decomposition. First, random monodromy loops [10] aim to determine subsets of points in W

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contained in the same irreducible component. This relies on the fact that the set of smooth points of an irreducible algebraic set is connected.

Second, given $Z \subset W$, the trace test of [11] is used to verify that Z is a witness point set for some algebraic set, i.e., there exists $\mathcal{J} \subset \{1, \ldots, m\}$ such that

$$Z = \bigcup_{j \in \mathcal{J}} X_j \cap \mathcal{L}.$$

There are two ways to show that $|\mathcal{J}| = 1$, i.e., Z is a witness point set for an irreducible component. If Z was constructed as a result of using random monodromy loops, then each point in Z must lie on the same irreducible component. Another approach is to show that the trace test does not hold for any nonempty and proper subset of Z.

Since the trace test of [11] uses only path tracking, we will refer to this as the *tracking trace test*. We show that this test is the first in a family of three methods, which are based on the zeroth, first, and second derivatives, respectively. The third of these methods, which is built on computing second derivatives, computes these derivatives locally at each point in Z and hence we call it a *local trace test*.

The remainder is organized as follows. In Section 2, we describe linear traces in numerical algebraic geometry and present three computational approaches. Section 3 considers the extension to parameterized algebraic sets. We demonstrate the methods on two examples in Section 4 and conclude in Section 5.

2 Trace test

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system with $X \subset \mathcal{V}(f) \subset \mathbb{C}^N$ a pure k-dimensional set. Let $\ell : \mathbb{C}^N \to \mathbb{C}^k$ be a general linear system with $\mathcal{L} = \mathcal{V}(\ell)$ and $W = X \cap \mathcal{L}$. If $X_1, \ldots, X_m$ are the irreducible components of X, the goal is to partition W into the m sets $W_1 = X_1 \cap \mathcal{L}, \ldots, W_m = X_m \cap \mathcal{L}$.

We first reduce to the multiplicity-one case as follows. Since the deflation sequence [9] with respect to f is the same for each $w \in W_i = X_i \cap \mathcal{L}$, we can first partition W based on deflation sequences. So, without loss of generality, we may assume that each point in W has the same deflation sequence. In particular, a byproduct of this computation is a polynomial system, which without loss of generality we call f, such that each irreducible component has multiplicity one.

We next reduce to the "square" case using Bertini's Theorem (see, e.g., [12, Thm. A.8.7] and [3, Thm. 9.3]). In particular, for a general $U \in \mathbb{C}^{(N-k)\times n}$, each X_i is an irreducible component of $\mathcal{V}(U \cdot f)$. Hence, without loss of generality, we may assume that $f : \mathbb{C}^N \to \mathbb{C}^{N-k}$ such that $X \subset \mathcal{V}(f)$ is pure k-dimensional and each irreducible component of X has multiplicity one with respect to f.

Suppose that $d = \deg X = |W|$ and $W = \{w_1, \ldots, w_d\}$. For a general $v \in \mathbb{C}^k$, consider the family of parallel slices $\mathcal{M}_t = \mathcal{V}(\ell - t \cdot v)$ for $t \in \mathbb{C}$, so that $\mathcal{M}_0 = \mathcal{L} = \mathcal{V}(\ell)$. For $i = 1, \ldots, d$, consider the paths $x_i(t)$ defined by

$$x_i(t) \in X \cap \mathcal{M}_t \quad \text{and} \quad x_i(0) = w_i. \tag{1}$$

The following forms the basis of traces in numerical algebraic geometry.

Theorem 1 ([11]). *With the setup as above, let $\mathcal{I} \subset \{1, \ldots, d\}$ be nonempty with $Z = \{w_i \mid i \in \mathcal{I}\} \subset W$. Then, there exists $\mathcal{J} \subset \{1, \ldots, m\}$ such that*

$$Z = \bigcup_{j \in \mathcal{J}} X_j \cap \mathcal{L}$$

if and only if

$$\mathrm{tr}_{\mathcal{I}}(t) = \sum_{i \in \mathcal{I}} x_i(t) \text{ is a vector of linear functions of } t. \tag{2}$$

This theorem can be used to create a trace test for images of algebraic sets using pseudowitness sets [8] (we consider a simple coordinate projection in Section 4.1), and for multihomogeneous witness sets [7].

Example 1. Consider the parabola $X = \mathcal{V}(f)$ where $f(\alpha, \beta) = \beta - \alpha^2$. For illustrative purposes, we consider the $\mathcal{L} = \mathcal{V}(\ell)$ where $\ell(\alpha, \beta) = 2\alpha + \beta - 3$ and take $v = \sqrt{-1}$. If $W = X \cap \mathcal{L} = \{w_1, w_2\} = \{(-3, 9), (1, 1)\}$, then

$$\mathrm{tr}_{\{1\}}(t) = \begin{bmatrix} -1 - \sqrt{4 + t\sqrt{-1}} \\ 5 + t\sqrt{-1} + 2\sqrt{4 + t\sqrt{-1}} \end{bmatrix} \text{ and } \mathrm{tr}_{\{2\}}(t) = \begin{bmatrix} -1 + \sqrt{4 + t\sqrt{-1}} \\ 5 + t\sqrt{-1} - 2\sqrt{4 + t\sqrt{-1}} \end{bmatrix}$$

are not linear in t, whereas

$$\mathrm{tr}_{\{1,2\}}(t) = \begin{bmatrix} 0 \\ 2\sqrt{-1} \end{bmatrix} \cdot t + \begin{bmatrix} -2 \\ 10 \end{bmatrix}$$

is indeed linear in t confirming that X is irreducible of degree 2.

Example 2. For each subsequent method, we will use the twisted cube curve

$$X = \{(s, s^2, s^3) \mid s \in \mathbb{C}\} \subset \mathbb{C}^3.$$

For illustrative purposes, we consider

$$f(\alpha, \beta, \gamma) = \begin{bmatrix} \beta - \alpha^2 \\ \gamma - \alpha^3 \end{bmatrix}, \quad \ell(\alpha, \beta, \gamma) = 2\alpha - 3\beta - \gamma + 2, \quad \text{and} \quad v = 1.$$

With $W = X \cap \mathcal{V}(\ell)$ where $|W| = 3$ and $\mathcal{I} = \{1, 2, 3\}$, Newton's identities yield

$$\mathrm{tr}_{\mathcal{I}}(t) = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \cdot t + \begin{bmatrix} -3 \\ 13 \\ -39 \end{bmatrix} \tag{3}$$

which is linear in t.

The following three tests determine if $\mathrm{tr}_{\mathcal{I}}(t)$ is a linear function, i.e., there exists $a, b \in \mathbb{C}^N$ such that $\mathrm{tr}_{\mathcal{I}}(t) = a \cdot t + b$. They are derived using the fact that $\mathrm{tr}_{\mathcal{I}}(t)$ is linear if and only if $\dot{\mathrm{tr}}_{\mathcal{I}}(t)$ is constant if and only if $\ddot{\mathrm{tr}}_{\mathcal{I}}(t)$ is zero corresponding with the zeroth, first, and second derivatives of $\mathrm{tr}_{\mathcal{I}}(t)$. The zeroth derivative trace test is the tracking test of [11].

2.1 Zeroth derivative trace test

The tracking trace test of [11] determines if $\mathrm{tr}_{\mathcal{I}}(t)$ is a linear function by evaluating it at 3 distinct sufficiently general values of t, say $t_1, t_2, t_3 \in \mathbb{C}$ facilitated by path tracking. Due to the genericity of $\mathcal{L}$ and v, one could take $t_1 = 0$, $t_2 = 1$, and $t_3 = -1$. That is, one needs to compute

$$\mathrm{tr}_{\mathcal{I}}(t_j) = \sum_{i \in \mathcal{I}} x_i(t_j)$$

where $x_i(t)$ defined in (1) are solution curves of $H : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ with

$$H(x,t) = \begin{bmatrix} f(x) \\ \ell(x) - t \cdot v \end{bmatrix} = 0. \tag{4}$$

With this setup, $\mathrm{tr}_{\mathcal{I}}(t)$ is linear in t if and only if

$$\frac{\mathrm{tr}_{\mathcal{I}}(t_2) - \mathrm{tr}_{\mathcal{I}}(t_1)}{t_2 - t_1} = \frac{\mathrm{tr}_{\mathcal{I}}(t_3) - \mathrm{tr}_{\mathcal{I}}(t_1)}{t_3 - t_1} = \frac{\mathrm{tr}_{\mathcal{I}}(t_3) - \mathrm{tr}_{\mathcal{I}}(t_2)}{t_3 - t_2}.$$

In the linear case, $\mathrm{tr}_{\mathcal{I}}(t) = a \cdot t + b$ where

$$a = \frac{\mathrm{tr}_{\mathcal{I}}(t_2) - \mathrm{tr}_{\mathcal{I}}(t_1)}{t_2 - t_1} \quad \text{and} \quad b = \mathrm{tr}_{\mathcal{I}}(t_1) - a \cdot t_1.$$

Example 3. With the setup from Ex. 2 and $t_1 = 0$, $t_2 = 1$, and $t_3 = -1$, the following table lists approximations of $x_i(t_j)$ for $i = 1, 2, 3$ and $j = 1, 2, 3$:

	$t_1 = 0$	$t_2 = 1$	$t_3 = -1$
	1.0000	0.8342	1.1284
$x_1(t_j)$	1.0000	0.6960	1.2733
	1.0000	0.5806	1.4368
	−0.5858	−0.3434	−0.7984
$x_2(t_j)$	0.3431	0.1179	0.6374
	−0.2010	−0.0405	−0.5089
	−3.4142	−3.4909	−3.3301
$x_3(t_j)$	11.6569	12.1861	11.0893
	−39.7990	−42.5401	−36.9280

so that

$$\mathrm{tr}_{\mathcal{I}}(0) = \begin{bmatrix} -3 \\ 13 \\ -39 \end{bmatrix}, \quad \mathrm{tr}_{\mathcal{I}}(1) = \begin{bmatrix} -3 \\ 13 \\ -42 \end{bmatrix}, \quad \mathrm{tr}_{\mathcal{I}}(-1) = \begin{bmatrix} -3 \\ 13 \\ -36 \end{bmatrix} \tag{5}$$

which one can use to easily recover (3).

2.2 First derivative trace test

Since $\mathrm{tr}_{\mathcal{I}}(t)$ is linear if and only if $\dot{\mathrm{tr}}_{\mathcal{I}}(t)$ is constant, this can be decided by evaluating $\dot{\mathrm{tr}}_{\mathcal{I}}(t)$ at 2 distinct sufficiently general values of t, say $t_1, t_2 \in \mathbb{C}$, facilitated by path tracking and derivative computations. Due to the genericity of $\mathcal{L}$ and v, one could take $t_1 = 0$ and $t_2 = 1$. Due to the relationship between the paths $x_i(t)$ in (1) and the homotopy $H(x,t)$ in (4),

$$\dot{x}_i(t) = -J_x H(x_i(t), t)^{-1} \cdot J_t H(x_i(t), t) = \begin{bmatrix} Jf(x_i(t)) \\ J\ell(x_i(t)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{6}$$

with corresponding Jacobian matrices $J_xH(x,t)$, $J_tH(x,t)$, $Jf(x)$, and $J\ell(x)$ so

$$\dot{\mathrm{tr}}_{\mathcal{I}}(t) = \sum_{i\in\mathcal{I}} \dot{x}_i(t) = \sum_{i\in\mathcal{I}} \begin{bmatrix} Jf(x_i(t)) \\ J\ell(x_i(t)) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

Therefore, $\mathrm{tr}_{\mathcal{I}}(t)$ is a linear function of t if and only if

$$\dot{\mathrm{tr}}_{\mathcal{I}}(t_1) = \dot{\mathrm{tr}}_{\mathcal{I}}(t_2) = \frac{\mathrm{tr}_{\mathcal{I}}(t_2) - \mathrm{tr}_{\mathcal{I}}(t_1)}{t_2 - t_1}.$$

In the linear case, $\mathrm{tr}_{\mathcal{I}}(t) = a \cdot t + b$ where

$$a = \dot{\mathrm{tr}}_{\mathcal{I}}(t_1) \quad \text{and} \quad b = \mathrm{tr}_{\mathcal{I}}(t_1) - a \cdot t_1.$$

Thus, the first derivative trace test replaces evaluating $\mathrm{tr}_{\mathcal{I}}(t_3)$, a path tracking computation, with evaluating $\dot{\mathrm{tr}}_{\mathcal{I}}(t_1)$ and $\dot{\mathrm{tr}}_{\mathcal{I}}(t_2)$, a linear algebra computation. We emphasize here that finding $\dot{\mathrm{tr}}_{\mathcal{I}}(t_1)$ does involve path tracking, but the cost incurred due to tracking paths is half that of the zeroth derivative trace test.

Example 4. With the setup from Ex. 2, we consider $t_1 = 0$ and $t_2 = 1$ with the values of $x_i(t_j)$ listed in Ex. 3. The following table lists approximations of the six values of $\dot{x}_i(t_j)$ for $i = 1, 2, 3$ and $j = 1, 2$ computed using (6):

	$t_1 = 0$	$t_2 = 1$
	-0.1429	-0.1963
$\dot{x}_1(t_j)$	-0.2857	-0.3276
	-0.4286	-0.4099
	0.2230	0.2698
$\dot{x}_2(t_j)$	-0.2612	-0.1853
	0.2295	0.0954
	-0.0801	-0.0735
$\dot{x}_3(t_j)$	0.5469	0.5129
	-2.8009	-2.6855

so that

$$\dot{\mathrm{tr}}_{\mathcal{I}}(0) = \dot{\mathrm{tr}}_{\mathcal{I}}(1) = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \tag{7}$$

which, together with $\mathrm{tr}_{\mathcal{I}}(0)$ in (5), one can easily recover (3).

2.3 Second derivative trace test

Since $\mathrm{tr}_{\mathcal{I}}(t)$ is linear if and only if $\ddot{\mathrm{tr}}_{\mathcal{I}}(t) \equiv 0$, this can be decided by evaluating $\ddot{\mathrm{tr}}_{\mathcal{I}}(t)$ at a sufficiently general $t_1 \in \mathbb{C}$ facilitated by derivative computations. Due to the genericity of $\mathcal{L}$ and v, we take $t_1 = 0$. Hence, $x_i(0) = w_i$ by (1) and $\dot{x}_i(0)$ as in (6) so that

$$\mathrm{tr}_{\mathcal{I}}(0) = \sum_{i\in\mathcal{I}} x_i(0) = \sum_{i\in\mathcal{I}} w_i \quad \text{and} \quad \dot{\mathrm{tr}}_{\mathcal{I}}(0) = \sum_{i\in\mathcal{I}} \dot{x}_i(0) = \sum_{i\in\mathcal{I}} \begin{bmatrix} Jf(w_i) \\ J\ell(w_i) \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

Due to the structure of $H(x,t)$ in (4), $\frac{\partial^2 H}{\partial x \partial t} = 0$ and

$$\ddot{x}_i(0) = -\begin{bmatrix} Jf(w_i) \\ J\ell(w_i) \end{bmatrix}^{-1} \cdot \begin{bmatrix} \dot{x}_i(0)^T \cdot \text{Hessian}(f_1)(w_i) \cdot \dot{x}_i(0) \\ \vdots \\ \dot{x}_i(0)^T \cdot \text{Hessian}(f_n)(w_i) \cdot \dot{x}_i(0) \\ 0 \end{bmatrix} \tag{8}$$

where $\text{Hessian}(f_j)(w_i)$ is the Hessian matrix of f_j evaluated at w_i. Hence, $\text{tr}_{\mathcal{I}}(t)$ is a linear function of t if and only if

$$\ddot{\text{tr}}_{\mathcal{I}}(0) = \sum_{i \in \mathcal{I}} \ddot{x}_i(0) = 0.$$

In the linear case, $\text{tr}_{\mathcal{I}}(t) = a \cdot t + b$ where

$$a = \dot{\text{tr}}_{\mathcal{I}}(0) \quad \text{and} \quad b = \text{tr}_{\mathcal{I}}(0).$$

Thus, the second derivative trace test replaces all path tracking with second derivative computations performed locally and hence we call it a *local trace test.*

Example 5. With the setup from Ex. 2, we consider $t_1 = 0$ with the values of $x_i(0)$ and $\dot{x}_i(0)$ listed in Ex. 3 and Ex. 4, respectively. Approximations of $\ddot{x}_i(0)$ for $i = 1, 2, 3$ computed using (8) are

$$\ddot{x}_1(0) = \begin{bmatrix} -0.0350 \\ -0.0292 \\ 0.0175 \end{bmatrix}, \quad \ddot{x}_2(0) = \begin{bmatrix} 0.0275 \\ 0.0671 \\ -0.1464 \end{bmatrix}, \quad \ddot{x}_3(0) = \begin{bmatrix} 0.0074 \\ -0.0380 \\ 0.1289 \end{bmatrix}$$

so that $\ddot{\text{tr}}_{\mathcal{I}}(0) = 0$. Thus, $\text{tr}_{\mathcal{I}}(0)$ and $\dot{\text{tr}}_{\mathcal{I}}(0)$ computed in (5) and (7) yield (3).

3 Parameterizations

In Section 2, we considered pure-dimensional $X \subset \mathcal{V}(f)$, i.e., X was contained in the solution set of $f = 0$. In this section, we consider pure-dimensional sets which arise as the image of an algebraic set under an algebraic map. For simplicity, we only consider $X = \overline{\{p(y) \mid y \in \mathbb{C}^k\}} \subset \mathbb{C}^N$ where $p : \mathbb{C}^k \to \mathbb{C}^N$ has rank k, i.e., rank $Jp(y) = k$ for generic $y \in \mathbb{C}^k$, as more general situations follow using similar computations. With this setup, X is irreducible with $\dim X = k$ so that the main question is to determine $\deg X$ via a trace test. That is, given $Z \subset W = X \cap \mathcal{L}$ where $\ell : \mathbb{C}^N \to \mathbb{C}^k$ is a general linear system and $\mathcal{L} = \mathcal{V}(\ell)$, one aims to decide if $Z = W$ so that $d = \deg X = |W| = |Z|$.

Since we are given $Z \subset W$, let $Z = \{z_1, \ldots, z_q\}$. Since $X = \overline{p(\mathbb{C}^k)}$ and $\mathcal{L}$ is general, we know that there exists $y_1, \ldots, y_q \in \mathbb{C}^k$ such that $z_i = p(y_i)$. Let $v \in \mathbb{C}^k$ be general and $\mathcal{M}_t = \mathcal{V}(\ell - t \cdot v)$. For each $i = 1, \ldots, q$, we consider the paths $x_i(t) \in X$ and $u_i(t) \in \mathbb{C}^k$ defined by

$$x_i(t) = p(u_i(t)) \in X \cap \mathcal{M}_t \quad \text{and} \quad u_i(0) = y_i. \tag{9}$$

In particular, $u_i(t) \in \mathbb{C}^k$ satisfies the k equations $\ell(p(u_i(t))) = t \cdot v$.

With $\mathcal{I} = \{1, \ldots, q\}$, the trace tests in Section 2 involve the computation of $\mathrm{tr}_{\mathcal{I}}(t) = \sum_{i=1}^{q} x_i(t)$, $\dot{\mathrm{tr}}_{\mathcal{I}}(t) = \sum_{i=1}^{q} \dot{x}_i(t)$, and $\ddot{\mathrm{tr}}_{\mathcal{I}}(t) = \sum_{i=1}^{q} \ddot{x}_i(t)$. Thus, all that remains is to compute $\dot{x}_i(t)$ and $\ddot{x}_i(t)$, namely

$$\dot{u}_i(t) = (J\ell(x_i(t)) \cdot Jp(u_i(t)))^{-1} \cdot v, \qquad \dot{x}_i(t) = Jp(u_i(t)) \cdot \dot{u}_i(t) \tag{10}$$

and

$$\ddot{x}_i(t) = \left(I - Jp(u_i(t)) \cdot (J\ell(x_i(t)) \cdot Jp(u_i(t)))^{-1} \cdot J\ell(x_i(t))\right) \begin{bmatrix} \dot{u}_i(t)^T \cdot \mathrm{Hessian}(p_1)(u_i(t)) \cdot \dot{u}_i(t) \\ \vdots \\ \dot{u}_i(t)^T \cdot \mathrm{Hessian}(p_N)(u_i(t)) \cdot \dot{u}_i(t) \end{bmatrix} \tag{11}$$

where $I \in \mathbb{C}^{N \times N}$ is the identity matrix.

Example 6. We again illustrate using the twisted cubic curve from Ex. 3 using the same ℓ and v. In this case, we have $x_i(t_j) = p(u_i(t_j))$ where $u_i(t_j) = (x_i(t_j))_1$, i.e., the first coordinate, and $p(s) = \left[s, s^2, s^3\right]^T$. Via (10) and (11), we have $\dot{u}_i(t_j) = (2 - 6u_i(t_j) - 3u_i(t_j)^2)^{-1}$,

$$\dot{x}_i(t_j) = \dot{u}_i(t_j) \cdot \begin{bmatrix} 1 \\ 2u_i(t_j) \\ 3u_i(t_j)^2 \end{bmatrix}, \quad \text{and} \quad \ddot{x}_i(t_j) = \dot{u}_i(t_j)^3 \cdot \begin{bmatrix} 6u_i(t_j) + 6 \\ 6u_i(t_j)^2 + 4 \\ 12u_i(t_j) - 18u_i(t_j)^2 \end{bmatrix}$$

which produces the values listed in the tables in Ex. 4 and Ex. 5.

4 Examples

The following compares the zeroth, first, and second derivative trace tests on two large examples. These examples utilized `Bertini` [2] for the path tracking and used Python with NumPy [13] to perform the linear algebra computations. For simplicity in our comparison, we utilize serial computations for all three trace tests but note that all three tests could be easily parallelized.

4.1 A curve from kinematics

One problem solved in [4] is the so-called 8 path-point synthesis problem for four-bar linkages derived from classical work of [1, 5]. That is, one aims to compute all four-bar planar linkages whose coupler curve passes through 8 given general points in the plane. Since one can freely set the orientation at one point, the 8 path-point synthesis problem is the one-pose and 7 path-point Alt-Burmester problem solved in [4] which defines a curve in $\mathbb{C}^8$ of degree 10,858. Following [4], we consider the system of 21 polynomials $f(x, y)$ where $x \in \mathbb{C}^8$ and $y \in \mathbb{C}^{14}$. Since the curve of interest is the natural projection of a solution curve in $\mathcal{V}(f)$ into $\mathbb{C}^8$, following [8], we take $\ell(x)$ as a random linear polynomial and $v \in \mathbb{C}$ random.

With this setup and $\mathcal{I} = \{1, \ldots, 10858\}$, we used the zeroth, first, and second derivative trace tests from Section 2 to verify that the degree of this curve in $\mathbb{C}^8$ is indeed 10,858 by showing that the first 8 coordinates of $\mathrm{tr}_{\mathcal{I}}(t)$ are linear in t. Using serial computations on an Intel Core i7, the zeroth derivative trace test took 21.6 minutes, the first derivative test took 9.5 minutes, and the second derivative test took 1.4 minutes.

4.2 A secant variety

In order to consider the border rank of the tensor corresponding to 2×2 matrix multiplication, the secant variety $X = \sigma_6(\mathbb{C}^4 \times \mathbb{C}^4 \times \mathbb{C}^4) \subset \mathbb{C}^{64}$ was considered in [6] which showed that $\dim X = 60$ and $\deg X = 15{,}456$. After selecting 60 random linear polynomials ℓ and random $v \in \mathbb{C}^{60}$, we performed the zeroth, first, and second derivative trace tests based on parameterizations in Section 3 which verified that the degree is indeed 15,456. Using serial computations on an AMD Opteron 6378 processor, the zeroth derivative test took 84.1 hours, the first derivative test took 42.2 hours, and the second derivative test took 0.2 hours. The vast difference in computation time is due to the use of adaptive precision during tracking; larger systems such as this, having 60 variables, often require higher precision than hardware types provide.

5 Conclusion

Decomposition of a pure-dimensional algebraic set into its irreducible components is fundamental to computational algebraic geometry. In numerical algebraic geometry, the pure-dimensional decomposition is performed using random monodromy loops verified by a trace test. By replacing path tracking with local derivative computations, we have developed a local trace test which examples show is computationally advantageous. Due to these results, we are in the process of developing a robust, high-performance, and parallel implementation.

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